

# Aspects of Bifurcation Theory for Piecewise-Smooth, Continuous Systems

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## Abstract

Systems that are not smooth can undergo bifurcations that are forbidden in smooth systems. We review some of the phenomena that can occur for piecewise-smooth, continuous maps and flows when a fixed point or an equilibrium collides with a surface on which the system is not smooth. Much of our understanding of these cases relies on a reduction to piecewise linearity near the border-collision. We also review a number of codimension-two bifurcations in which nonlinearity is important.

**Keywords:** bifurcation, border-collision, discontinuous bifurcation, saddle-node, Hopf, Neimark-Sacker

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## 1. Introduction

A dynamical system is *piecewise smooth* (PWS) if its phase space can be partitioned into countably many regions within which it is smooth. The codimension-one sets on which the dynamics is not smooth are called *switching manifolds*. The recent explosion of interest in these systems is demonstrated by the rapid growth of publications in the past two decades shown in Fig. 1, the appearance of a number of texts [1, 2, 3, 4, 5, 6, 7], as well as this issue of Physica D.

Classical bifurcation theory relies heavily on the assumption of a certain degree of smoothness in the singularity conditions (e.g. zero eigenvalues), the nondegeneracy requirements (nonvanishing coefficients of some terms in a power series), and the use of the center manifold theorem (to achieve a dimension reduction). Bifurcations can occur in nonsmooth systems that are forbidden in smooth systems; those that occur due to the interaction of invariant sets with a switching manifold are known as *discontinuity induced bifurcations*. A simple example corresponds to the PWS continuous, one-dimensional family of “tent maps”,

$$T(x; r) = \begin{cases} rx, & x < \frac{1}{2} \\ r(1-x), & x \geq \frac{1}{2} \end{cases}, \quad (1)$$

for which infinitely many periodic orbits are created as  $r$  increases through 1, without the usual sequence of period doubling bifurcations of smooth maps [8, 9].<sup>3</sup>

Of the many types of discontinuities that can occur, perhaps the simplest is a discontinuity in the first derivative; in this case the dynamical system is piecewise-smooth and continuous (PWSC). The simplest discontinuity induced bifurcations in these systems correspond to an equilibrium (for ODEs) or a fixed point (for maps) that lies on the switching

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<sup>3</sup>The general piecewise-linear, one-dimensional map was treated by Nusse and collaborators [10] in their pioneering papers on border-collision bifurcations, see also [6].

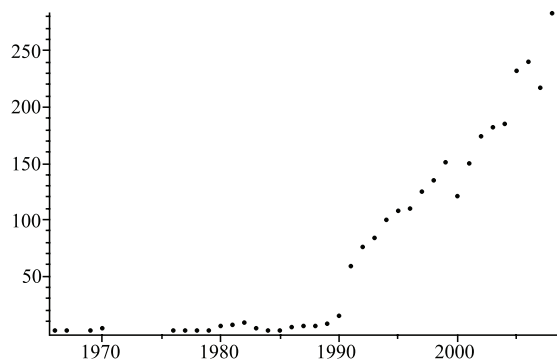


Figure 1: Number of papers published each year on piecewise smooth dynamics from a search of the ISI Web of Knowledge® database. Keywords were Topic=(piecewise smooth) OR (discontinuity induced) OR (nonsmooth) OR (grazing bifurcation) OR (border-collision bifurcation) AND Topic=(dynamics).

manifold. The resulting bifurcations are known as *discontinuous* or *border-collision* bifurcations, respectively. The bifurcations occurring in this class of systems are reviewed in this paper.

More extreme discontinuities include *hybrid systems* in which the dynamics may be defined by different types of models in different regions [11, 12], and *Filippov systems* where the vector field is discontinuous on a switching manifold and the behavior may include sliding along the manifold [13].

In this paper we will review some recent results on low-codimension discontinuous and border-collision bifurcations when the switching manifold is smooth at the bifurcation point.<sup>4</sup> One of the difficulties in treating these bifurcations generally corresponds to the absence of a center manifold reduction. Even though there are still many open questions, there are a number of cases of  $N$ -dimensional flows and maps for which general bifurcation results have been obtained.

We focus on PWSC systems whose components are  $C^k$  (for some  $k \in \mathbb{N}$ ). With this restriction we disregard, for instance, maps with square-root terms that arise from regular grazing in Filippov systems [15, 6]; however, we are able to exploit a major simplification: border-collision bifurcations can often be treated with a piecewise-linear model. For example, nonlinear terms do not influence generic discontinuous saddle-node-like and Hopf-like bifurcations. An important consequence is that the size of invariant sets created in these bifurcations grows linearly with parameters instead of as some fractional power for the smooth case.

Piecewise-smooth dynamics arises in many areas including electronics [4], control theory [16], impacting mechanical systems [5, 17], economics [18, 19, 20] and biology [21, 22, 23]—an extensive set of references can be found in [6, 7]. While continuous models are not always appropriate for these applications, there are many cases for which the model is given by either a continuous vector field or map. For example, PWSC dynamics occurs in the well-known Chua circuit, introduced in 1983, that contains a diode often modeled with a piecewise-linear, current-voltage response curve [24]. Similarly power circuitry like DC/DC buck/boost converters, used to decrease or increase voltage levels in common devices such as cell phones, can often be modeled by PWSC maps [25]. Piecewise smooth, continuous economic models include discrete-time models for trade [26], production-distribution [27] and of a socialist economy [28]. Mechanical systems with impacts, or with transitions from static to sliding friction provide another large application area. Continuous models often arise in these systems when a body grazes a rigid surface, leading to so-called grazing bifurcations [15, 29]. Biological models that are PWSC include models of neurons [30], neural networks [31], and cybernetic models which reduce complex chemical models by selecting optimal reaction pathways [32].

In the next section we introduce equations ideal for analyzing local bifurcations of PWSC systems. We solve for the equilibria or fixed points for these equations and describe the observer canonical form. Section 3 summarizes recent

<sup>4</sup>Bifurcations at corners of a switching manifold are discussed in [5, 14].

results regarding codimension-one and two bifurcations for PWSC vector fields. A discussion for the discrete-time scenario is presented in §4.

## 2. Local Framework

Near a sufficiently smooth, codimension-one switching manifold one can always choose coordinates so that the manifold is represented by the vanishing of one of the coordinates, say the first coordinate [33]. To simplify the notation, for any point  $x \in \mathbb{R}^N$  we let

$$s = e_1^\top x \quad (2)$$

denote the first element of  $x$ ; consequently, the switching manifold is the plane  $s = 0$ . In this case a PWSC dynamical system takes different forms in the left-half space,  $s < 0$ , and the right-half space,  $s > 0$ :

$$f(x; \xi) = \begin{cases} f^{(\text{L})}(x; \xi), & s \leq 0 \\ f^{(\text{R})}(x; \xi), & s \geq 0 \end{cases} . \quad (3)$$

Here the functions  $f^{(i)} : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^M$ ,  $i \in \{\text{L}, \text{R}\}$  are  $C^k$  functions on the  $N$ -dimensional phase space with coordinates  $x$  and parameters  $\xi \in \mathbb{R}^M$ . We will assume that  $k \geq 1$ , though often the assumed  $k$  will need to be larger. The function  $f$  may represent a vector field, in which case the dynamics is the system of ODEs

$$\dot{x} = f(x; \xi) , \quad (4)$$

or a map, in which case the dynamics is the discrete-time system

$$x \mapsto x' = f(x; \xi) . \quad (5)$$

An orbit in the discrete case is a sequence  $\{x_t : x_{t+1} = f(x_t, \xi), t \in \mathbb{Z}\}$ . We will discuss these separately in §3 and §4, respectively.

For the simplest bifurcations, we assume that the system has an equilibrium or fixed point on the switching manifold at some specific parameter value. Without loss of generality this point can be taken to be the origin, and we can write the parameters as  $\xi = (\mu, \eta)$  where  $\mu \in \mathbb{R}$  represents a parameter that vanishes at the border collision. Then we can write

$$f^{(i)}(x; \mu, \eta) = \mu b(\mu, \eta) + A_i(\mu, \eta)x + O(|x|^2) + o(k) , \quad (6)$$

where  $A_{\text{L}}$  and  $A_{\text{R}}$  are the  $N \times N$  Jacobian matrices at the origin. Under the assumption of continuity the two vector fields must agree whenever  $s = 0$ , which implies that all of the columns of these two matrices must be identical except for the first

$$A_{\text{L}} e_i = A_{\text{R}} e_i , \quad i \neq 1 .$$

The parameter  $\mu$  corresponds to the primary parameter that unfolds a simple discontinuous or border-collision bifurcation, while  $\eta$  will be used to represent parameters unfolding higher codimension cases. For simplicity, we suppress the dependence of the system on  $\eta$ , though the discussion will refer to these additional parameters as needed.

The nonlinear terms in (6) can be regarded as higher-order in the sense that any structurally stable dynamics of the piecewise-linear system will persist when the nonlinear terms are added. When  $\mu \ll 1$ , it is useful to consider a scaled system, defining  $z = \frac{x}{|\mu|}$  when  $\mu \neq 0$  and  $z = x$  when  $\mu = 0$  to obtain

$$\dot{z} = \begin{cases} \sigma b(0) + A_{\text{L}}(0)z, & s < 0 \\ \sigma b(0) + A_{\text{R}}(0)z, & s > 0 \end{cases} , \quad (7)$$

where  $\sigma = \text{sgn}(\mu)$  (and is 0 when  $\mu = 0$ ). A nice feature of (7) is that  $\mu$  has been replaced by the discrete parameter  $\sigma \in \{-1, 0, 1\}$ . Note that any structurally stable, bounded invariant set of (7) will approximate an invariant set of (3) as  $\mu$  tends to zero, and moreover that its size (in  $x$ ) will shrink linearly to zero. This simple feature is one of the most prominent characteristics of discontinuous and border-collision bifurcations.

If the matrices  $A_i(0)$  are nonsingular, the implicit function theorem implies that the ODE (4) has two potential equilibria,

$$x^{*(i)}(\mu) = -\frac{\varrho^\top(0)b(0)}{\det(A_i(0))}\mu + O(\mu^2), \quad (8)$$

where

$$\varrho^\top = e_1^\top \text{adj}(A_L(\mu)) = e_1^\top \text{adj}(A_R(\mu)), \quad (9)$$

and  $\text{adj}(A)$  is the adjugate of  $A$  (defined by  $A \text{adj}(A) = \det(A)I$  and extended by continuity for  $\det(A) = 0$ ). Note that (8) also pertains to the map (5) if we replace  $A_i$  by  $I - A_i$  [7].

The equilibria (8) are said to be *admissible* when  $s^{*(L)} \leq 0$  and  $s^{*(R)} \geq 0$ . Otherwise they are *virtual*. Even virtual equilibria play a prominent role in the dynamics of (7), since the eigenspaces of such equilibria typically intersect the admissible half-space and these intersections are therefore (forward or backward) invariant sets of the full system.

A simple consequence of (8) was first noted by Mark Feigin [34] (for the map case): there are two distinct types of border collision in this nonsingular case: if  $\det(A_L(0))$  and  $\det(A_R(0))$  (or  $\det(I - A_L(0))$  and  $\det(I - A_L(0))$  for the map case) have the same sign, then the equilibria (8) are admissible for different signs of  $\mu$ , so that near the origin the system (3) has a unique equilibrium. On the other hand if the signs of the Jacobian determinants differ, then the equilibria are admissible for only one sign of  $\mu$  and they collide and annihilate as  $\mu \rightarrow 0$ , giving rise to a *nonsmooth fold* bifurcation.

Additional coordinate transformations can be performed on (7) to reduce the number of parameters. Normally one could hope to transform the matrices to Jordan form; however, this cannot always be done simultaneously for both matrices, nor would it typically maintain the definition of the switching manifold,  $s = 0$ . However, under the condition of *observability*, the matrices can be reduced to *companion form*

$$C = \left[ \begin{array}{c|c} -\Delta & I \\ \hline & 0 \end{array} \right],$$

where  $\Delta$  is a vector and  $I$  is the  $(N - 1) \times (N - 1)$  identity matrix. Note that the components of  $\Delta$  are the coefficients of the characteristic polynomial  $p(\lambda) = \det(\lambda I - C)$ . Consequently every matrix has a unique companion form.

**Theorem 1** (Observer Canonical Form [6, 7]). *There is a coordinate transformation that reduces (3) with (6) to the observer canonical form: (3) with*

$$\hat{f}^{(i)}(x; \mu) = \mu e_N + C_i(\mu)x + O(|x|^2) + o(k), \quad (10)$$

for small  $\mu$  and where  $C_i$  is the companion matrix of  $A_i$ , if and only if  $A_L(0)$  has no eigenspace orthogonal to  $e_1$ .

Note that the switching manifold for the observer form is still  $s = 0$ . The piecewise-linear version of (10) has  $N$  parameters for each matrix, so its complete bifurcation picture is determined by  $2N$  parameters, in addition to  $\mu$ .

### 3. Continuous-Time Systems

The determination of the stability of an equilibrium located on a switching manifold is an important and fundamental yet difficult problem in nonsmooth systems [22, 35]. For PWSC systems, stability depends on the eigenspaces of each adjoining smooth subsystem and their relative configuration with respect to switching manifold(s), and, in special cases, nonlinear terms of the system [36, 37]. In the absence of nonlinear terms, radial symmetry allows for a projection of the dynamics onto a space of dimension one less than the dimension of the system, which is a useful simplification [38].

When there is a single switching manifold, as for (3), in one or two dimensions the stability problem is straightforward [39]. In three or more dimensions the problem is significantly more complicated. For example, an equilibrium on a switching manifold may be unstable even if all eigenvalues of both  $A_L$  and  $A_R$  have negative real part [40]. There may exist cone-shaped invariant sets akin to slow eigenspaces of smooth systems [41, 42]. By contrast, in two dimensions such an equilibrium undergoes a Hopf-like bifurcation when, roughly speaking, trajectories of the linearization

spiral inwards as much as they spiral outwards, per rotation [43, 44, 45]. These considerations are important for some models of real-world systems for which some natural constraint results in an equilibrium that remains on a switching manifold under parameter perturbation [42, 46].

Bifurcations of equilibria in smooth systems of arbitrary dimension are readily classifiable as saddle-node, Hopf, and so forth, because the degenerate dynamics occur on a low dimensional subspace (center manifold). For the PWSC system, (3), this dimension reduction cannot be performed because the system is not differentiable in a neighborhood of the equilibrium. For this reason discontinuous bifurcations resist a simple classification. Though discontinuous bifurcations may generate complex dynamics, such as Silnikov homoclinic chaos [47, 48], high-dimensional, PWSC systems often can exhibit only low-dimensional dynamics. It remains a difficult problem to develop a general theory of dimension reduction for such systems.

One idea is to look at the eigenvalues of the convex hull of the two linearizations of the equilibrium [5, 49]:

$$\hat{A} = \left\{ (1 - \sigma)A_L(0) + \sigma A_R(0) \mid \sigma \in [0, 1] \right\},$$

For  $\sigma \in [0, 1]$ , the eigenvalues of  $\hat{A}$  provide a continuous connection between the eigenvalues of  $A_L(0)$  and  $A_R(0)$  and are collectively referred to as the *eigenvalue path*. The idea is that a bifurcation occurs if and only if the eigenvalue path intersects the imaginary axis, though currently this remains a conjecture. Furthermore, examples indicate that the complexity of the discontinuous bifurcation increases with the number of intersections between the eigenvalue path and the imaginary axis. In particular, the presence of exactly one intersection typically yields a discontinuous bifurcation that is a natural analogue of a familiar smooth bifurcation. Nonsmooth counterparts of saddle-node, Hopf, pitchfork and transcritical bifurcations have been studied [5]. The main difference is that invariant sets created at nondegenerate discontinuous bifurcations grow in size linearly, to lowest order.

When (3) is planar, all possible codimension-one, discontinuous bifurcations that may occur have been determined [50, 39]. If the equilibria  $x^{*(L)}$  and  $x^{*(R)}$ , (8), are nondegenerate they may be classified as saddles, attracting nodes, attracting foci, repelling foci or repelling nodes. The bifurcations that result from a border collision between any pair of these equilibria are indicated in Fig. 2.

One possible case is that a collision of an equilibrium with a switching manifold yields no bifurcation in the sense that no invariant sets are created at the collision, although there is a piecewise-topological change in the sense of [6]. This occurs exactly when the equilibrium fails to change stability at the bifurcation, and is labeled “NB” in Fig. 2.

When one equilibrium is a saddle and the other is not, the two collide and annihilate in a saddle-node-like bifurcation, labeled “DSN” in the figure. An additional bifurcation may occur if the second equilibrium is a focus: a periodic orbit of stability opposite to the focus can be created at the discontinuous bifurcation [5, 7], this is labeled “DHBSN”.

The discontinuous bifurcation also gives rise to a periodic orbit if one equilibrium is attracting, the other is repelling and at least one is a focus, analogously to a Hopf bifurcation and so labeled “DHB”. When both are foci the stability of the generated orbit is the same as the weaker focus, where strength is measured by the radial factor by which the focus takes trajectories toward or away from the equilibrium upon a rotation of  $180^\circ$  [51, 52]. Thus the criticality of the bifurcation is determined by linear terms in the ODEs, in contrast to the smooth Hopf bifurcation.

The case that  $x^{*(L)}$  and  $x^{*(R)}$  are nodes of opposite stability is particularly unusual; we refer to the resulting bifurcation as a “stability-switching” bifurcation, labeled “SS” in the figure. No periodic orbit can be created locally due to the presence of the invariant slow manifolds of the nodes. In the absence of nonlinear terms it is useful to add a “circle at infinity” to the phase plane, an example is shown in Fig. 3. The resulting discontinuous bifurcation is then seen to be analogous to a heteroclinic bifurcation in that at the singularity there exists a continuous loop between the equilibrium on the switching manifold and two saddles at infinity. This connection is broken as soon as the equilibrium of the piecewise-linear system leaves the switching manifold. The presence of nonlinear terms may cause a periodic orbit to be created at the bifurcation; however, this orbit will not shrink to the equilibrium at the singularity. This discontinuous bifurcation is seen in, for instance, a piecewise-linear version of the FitzHugh-Nagumo model [53].

A variety of codimension-two, discontinuous bifurcations have been analyzed. For example, the solid lines in Fig. 2 correspond to codimension-two discontinuous bifurcations for planar systems; their unfoldings are described in [7]. Codimension-one discontinuous bifurcations occur along a curve in a two-parameter bifurcation diagram; an example taken from [32] is shown in Fig. 4. Curves of classical codimension-one, local bifurcations (such as saddle-

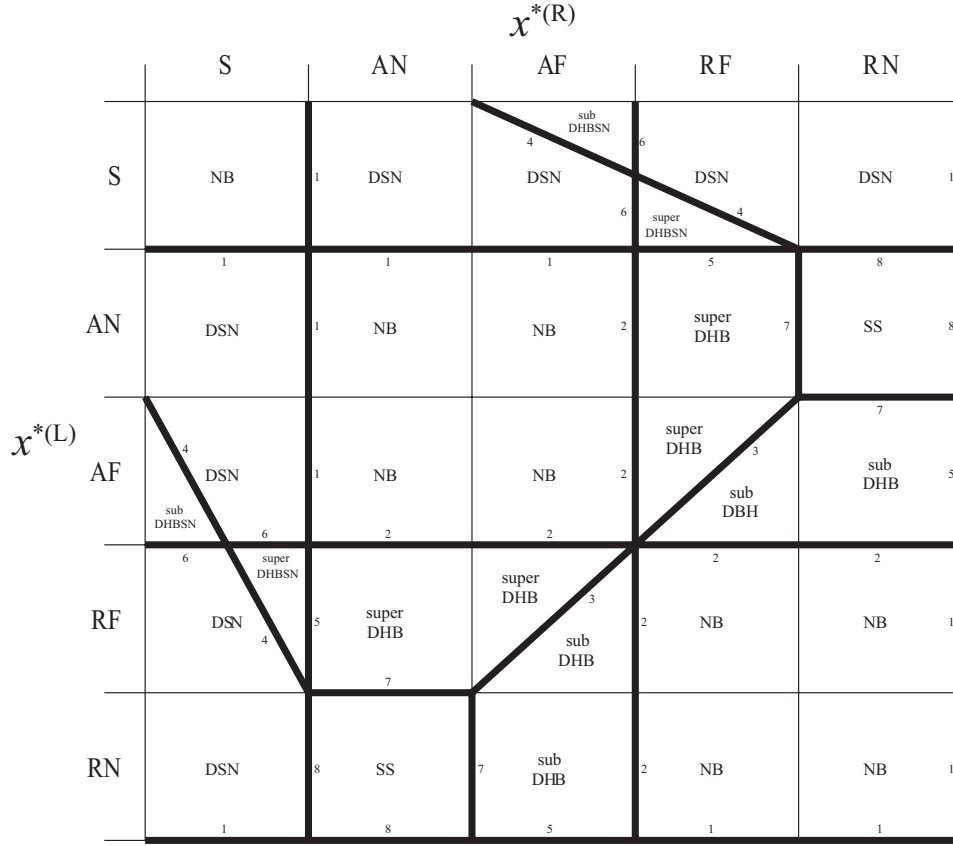


Figure 2: Codimension-one discontinuous bifurcations involving a single switching manifold for planar systems, taken from [7]. Each row [column] represents an equilibrium classification for  $f^{(L)}$  [ $f^{(R)}$ ], S - saddle; AN - attracting node; AF - attracting focus; RF - repelling focus; RN - repelling node. As in [7], names are given to the five distinct codimension-one phenomena: NB - no bifurcation; DSN - discontinuous saddle-node; DHB - discontinuous Hopf; DHBSN - discontinuous Hopf-saddle-node; SS - stability switching bifurcation. The chosen layout allows for a classification of some codimension-two scenarios—these are the thick lines in the figure. Since smooth parameter variation can produce a change from a saddle to a repelling node, the left and right sides of the figure should be considered coincident, as should the top and bottom, see [7] for further details.

node and Hopf bifurcations) involving an equilibrium of the discontinuous bifurcation may have an endpoint at the discontinuous bifurcation curve, like in Fig. 4, since beyond this curve the classical bifurcation is virtual. Roughly speaking, as one moves along a curve of saddle-node or Hopf bifurcations, the point in phase space at which the bifurcation occurs varies linearly. Equilibria created in a saddle-node bifurcation move apart from one another by an amount proportional to the square-root of the parameter change and for this reason the saddle-node bifurcation curve is tangent to the discontinuous bifurcation curve at its endpoint [32]. Similarly the amplitude of the periodic orbit created in a Hopf bifurcation grows proportional to the square-root of the parameter change, hence this Hopf cycle undergoes grazing along a curve tangent to the Hopf bifurcation locus at the codimension-two endpoint [32]. For planar systems, at the discontinuous bifurcation on one side of the codimension-two point a periodic orbit is created with stability determined by the linear terms of an appropriate piecewise-linear expansion. If this periodic orbit is of opposite stability to the Hopf cycle, then the two periodic solutions collide in a saddle-node bifurcation along a curve extremely close to the grazing locus; specifically, the series expansions of the two curves agree up to sixth order [54]. A rigorous analysis of this bifurcation in higher dimensions, as well as a multitude of other codimension-two scenarios, such as the simultaneous occurrence of a discontinuous bifurcation with either a pitchfork or a transcritical bifurcation, remains to be done.

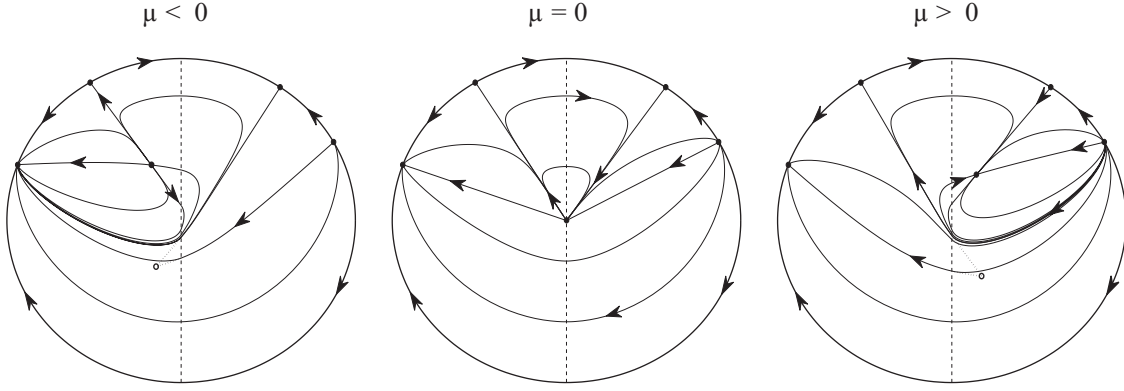


Figure 3: Phase portraits including a “circle at infinity” for a planar, piecewise-linear, continuous system undergoing a “stability-switching” bifurcation. The equilibria,  $x^{*(L)}$  and  $x^{*(R)}$ , (8), are nodes of opposite stability and indicated by hollow circles when virtual. When  $\mu > 0$ , the basin of attraction of  $x^{*(R)}$  is bounded by a heteroclinic connection between equilibria at infinity. The trajectory corresponding to the slow eigenvector of  $x^{*(L)}$  coincides with this connection for  $x < 0$ ; for  $x > 0$  this trajectory is virtual and shown by a dotted line.

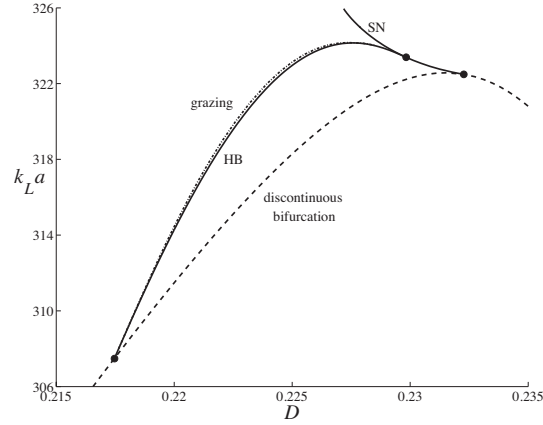


Figure 4: A two-parameter bifurcation diagram of a PWSC model of yeast growth, taken from [32]. Loci of classical saddle-node (SN) and Hopf (HB) bifurcations emanate from a locus of discontinuous bifurcations. The Hopf cycle grazes a switching manifold along the dash-dot curve near the Hopf locus.

#### 4. Discrete-Time Systems

As for continuous-time systems, the stability of fixed points of discrete-time, PWSC systems is typically not straightforward to determine. For example the stability of a fixed point at the origin of a planar, piecewise-linear map may be treated by reducing it to a circle map with a “dilation ratio” assigned to each point describing the factor by which the map takes the point towards or away from the origin [55]. The distribution of the dilation ratios over the natural measure of the circle map can reveal the stability of the origin. Of particular interest of late has been the scenario referred to as a *dangerous bifurcation* [56, 57, 58]. This corresponds to the case that the fixed point is stable on both sides of the border-collision bifurcation but is not stable at the bifurcation and may be analyzed by considering attractors at infinity [59].

In one-dimension, the map (3) may be written as

$$x' = \begin{cases} \mu + a_L x, & x \leq 0 \\ \mu + a_R x, & x \geq 0 \end{cases}, \quad (11)$$

and has dynamics that are summarized by Fig. 5, though it should be noted that unstable solutions are not indicated in this figure. The map (11) can exhibit discontinuous analogs of saddle-node and Hopf bifurcations for smooth maps, the primary difference being that invariant sets separate linearly. Furthermore, (11) can have a stable  $n$ -cycle (periodic solution of period  $n$ ) for any  $n$ , or a chaotic attractor. For further details see [10, 60, 61, 6].

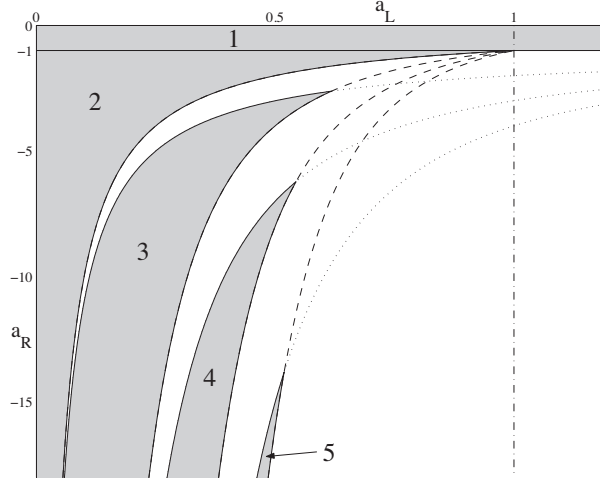


Figure 5: A two-parameter bifurcation diagram of the one-dimensional, piecewise-linear, continuous map (11), for  $\mu > 0$ , taken from [7]. Within each shaded region there exists a periodic solution of the indicated period. For  $a_L < 1$ , there exists a chaotic attractor within the white region.

Dynamics of piecewise-smooth maps are often easier to describe through the use of symbolics [62, 30, 63, 64]. Since here we are considering only dynamics near a single switching manifold, we consider sequences,  $\mathcal{S}$ , comprised of only two letters, L and R. Any orbit,  $\{x_i\}$ , of (3) with (6), may be assigned a symbol sequence:

$$\mathcal{S}_i = \begin{cases} \text{L} , & \text{if } s_i < 0 \\ \text{R} , & \text{if } s_i > 0 \end{cases} , \quad (12)$$

where  $s_i = e_1^\top x_i$  and no restriction is placed on  $\mathcal{S}_i$  if  $s_i = 0$ .

For the piecewise-linear case,  $\{x_0, x_1, \dots, x_{n-1}\}$  is an  $n$ -cycle if

$$x_0 = M_S x_0 + P_S b \mu ,$$

where

$$\begin{aligned} M_S &= A_{S_{n-1}} \dots A_{S_0} , \\ P_S &= I + A_{S_{n-1}} + A_{S_{n-1}} A_{S_{n-2}} + \dots + A_{S_{n-1}} \dots A_{S_1} , \end{aligned} \quad (13)$$

Following [65, 66, 67], if 1 is not an eigenvalue of the matrix  $M_S$ , then

$$x_0 = (I - M_S)^{-1} P_S b \mu ,$$

and

$$s_0 = \frac{\det(P_S)}{\det(I - M_S)} \varrho^\top b \mu .$$

where  $\varrho^\top = e_1^\top \text{adj}(I - A_L(\mu))$  (as implied by the discussion of (9), which is stated for continuous-time systems).

These algebraic methods can be used to find admissible periodic solutions, and hence to find regions in parameter space where periodic solutions exist for some fixed  $\mu$ . As in the smooth case, these regions are known as resonance or



Arnold tongues, see Fig. 6. Boundaries of resonance tongues correspond to the corresponding  $n$ -cycle losing stability or undergoing a border-collision. In the latter case one point of the  $n$ -cycle typically collides with the switching manifold. Remarkably a piecewise-linear expansion of the appropriate  $n^{\text{th}}$ -iterate of the map will also have the form (7), i.e., exactly the same form as the underlying map [60, 7]. Hence (7) may also be used to analyze border-collision bifurcations of  $n$ -cycles. The stable and unstable manifolds of periodic saddles may undergo homoclinic tangencies leading to homoclinic tangles and chaos [68].

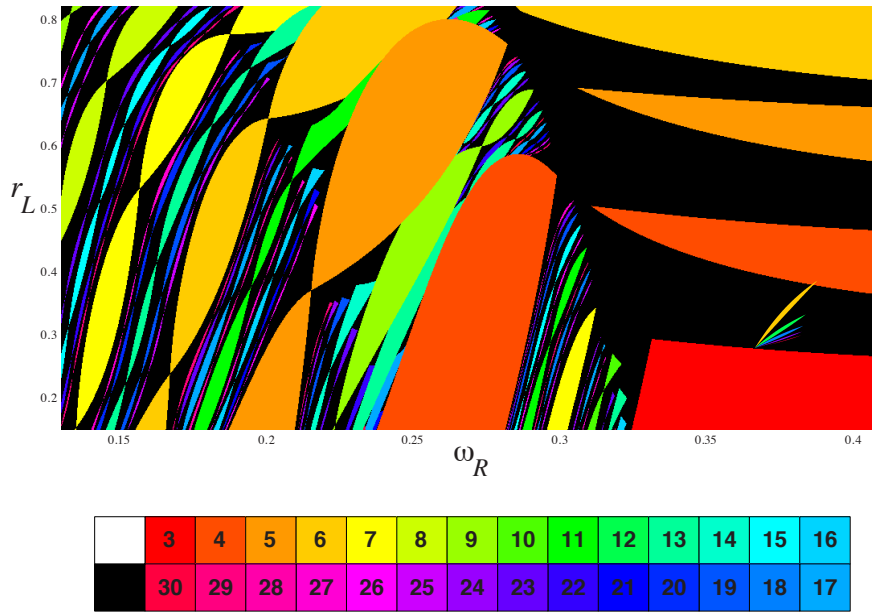


Figure 6: Resonance tongues of (7) with (10) for  $\mu > 0$  where the multipliers of  $C_L(0)$  and  $C_R(0)$  are  $r_L e^{\pm 2\pi i \omega_L}$  and  $\frac{1}{R} e^{\pm 2\pi i \omega_R}$ , respectively, and  $s_R = 0.8$ ,  $\omega_L = 0.09$ , taken from [7]. Period is indicated by the color bar. Within the black regions, forward orbits either approach an orbit of period greater than 30 or are aperiodic.

Resonance tongues in a generic two-parameter bifurcation diagram, like Fig. 6, for piecewise-linear, continuous maps often display a structure resembling a string of sausages [69, 70, 71, 18, 68, 65, 72]. A point at which a tongue has zero width is known as a *shrinking point*. When the symbol sequence of the periodic orbit is *rotational*, as defined in [66], the structure near a shrinking point is well understood. Rotational symbol sequences are natural as they correspond to rigid translations on an invariant circle. A rigorous unfolding of shrinking point bifurcations for rotational periodic orbits can be given under reasonable nondegeneracy conditions. The analysis shows that four distinct border-collision curves emanate from a shrinking point; each is tangent to one of the others at the shrinking point, an example is shown in Fig. 7. For a nonlinear, PWSC map, shrinking points occur only in the limit  $\mu \rightarrow 0$ . As  $\mu$  grows, they break apart through the formation of curves of classical saddle-node bifurcations, as shown in Fig. 8 [67]. The existence of quasiperiodic orbits with irrational rotation numbers can be inferred from the numerics as limits of periodic orbits [65]; however, these have not been treated rigorously as of yet, except in some special cases [73, 74].

The upper boundary of the period-three tongue (the red region in the lower right corner of Fig. 6) corresponds to a loss of stability of a 3-cycle when a multiplier becomes  $-1$ . At this boundary, a 6-cycle is created, though it is unstable and thus not shown in the figure, for further details see [75, 26, 65, 7]. Near the point  $(\omega_R, r_L) \approx (0.366, 0.278)$  small, higher-period resonance tongues emanate from a special point on the boundary of the period-three tongue in a manner that is only partially understood [65, 7].

If one of  $A_L(0)$  and  $A_R(0)$  for (3) has a multiplier on the unit circle, the border-collision bifurcation at  $\mu = 0$  is degenerate. The three codimension-two cases—a multiplier 1, a multiplier  $-1$ , or a complex pair of multipliers with unit modulus—correspond to the coincidence of a border-collision bifurcation with a saddle-node, a period-

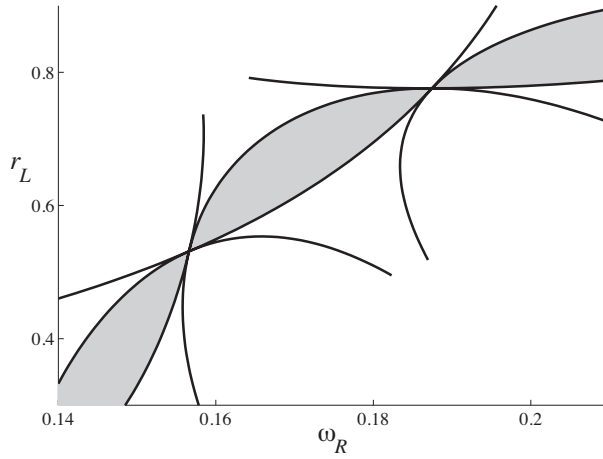


Figure 7: A magnification of the resonance tongue in Fig. 6 corresponding to a rotation number of  $1/7$ . The boundary of the resonance tongue is defined by several curves along which there exists a 7-cycle with one point on the switching manifold.

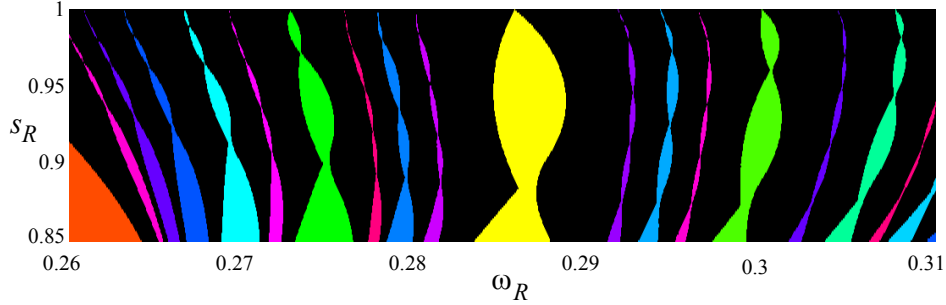


Figure 8: Resonance tongues for a PWSC map with multipliers and color scheme as in Fig. 6, taken from [67]. Here  $r_L = 0.2$ ,  $\omega_L = \omega_R$ ,  $\mu = 0.2$  and the map  $f^{(L)}$  has a quadratic nonlinearity. The right boundary of each resonance tongue has become  $C^1$  through the creation of a curve of saddle-node bifurcations, and the non-terminating shrinking points have been destroyed.

doubling, or a Neimark-Sacker bifurcation, respectively. Several unfoldings of these bifurcations are given in [7]. Recently these three scenarios have been unfolded by Colombo and Dercole [76] without the assumption that the map is continuous, in fact without any information about the map on one side of the border-collision bifurcation. The results mirror those described in §3 for the continuous-time case. With reasonable nondegeneracy assumptions, loci of saddle-node bifurcations and border-collision bifurcations intersect tangentially, whereas loci of period-doubling and Neimark-Sacker bifurcations intersect the border-collision bifurcation loci transversely. Since the newly created solutions along period-doubling and Neimark-Sacker bifurcation loci grow in size as the square-root of a parameter, they consequently undergo border-collision along a tangent curve as sketched Fig. 9. The nature of the bifurcation at this additional border-collision depends on the particular map. For example, it can be shown for the one-dimensional case that an attracting chaotic solution may be created along the period-doubling, border-collision curve [77].

## 5. Discussion

Dynamics related to the interaction of an equilibrium in a PWSC flow, or a fixed point in a PWSC map, with a smooth switching manifold are determined by a piecewise series expansion (3) with (6), and, except in special cases, only the linear terms of this expansion are important. Consequently invariant sets created at a discontinuous or border-collision bifurcation grow linearly, to lowest order, with respect to parameter change. Arguably this is

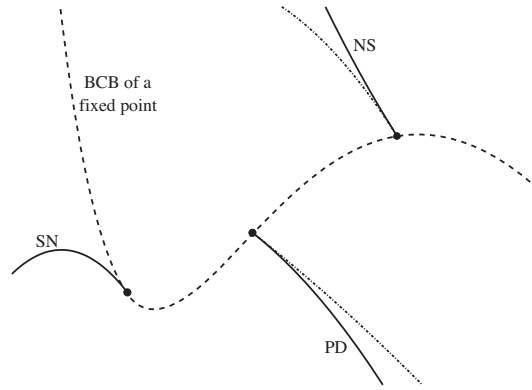


Figure 9: A schematic two-parameter bifurcation diagram for a PWSC map. The long dashed curve corresponds to a border-collision bifurcation of a fixed point. The solid curves correspond to loci of classical saddle-node (SN), period-doubling (PD) and Neimark-Sacker (NS) bifurcations. The dash-dot curves correspond to the collision of the period-doubled solution and the invariant circle with the switching manifold. The tangency or transversality of these curves follows from the general unfolding in [7, 76].

the most important feature that distinguishes discontinuity induced bifurcations with bifurcations of smooth systems. For example this difference has recently been found in a cardiac model [78, 23] suggesting that a border-collision bifurcation and not a classical period-doubling bifurcation generates the observed period-doubled solution.

Discontinuous bifurcations in one and two-dimensional systems are well understood, as are simple border-collision bifurcations. Often the local dynamics is determined by global properties of the appropriate piecewise-linear approximation. A number of codimension-two bifurcations that correspond to the coincidence of a discontinuous or border-collision bifurcation with one of several classical bifurcations have also been unfolded. However, several codimension-two scenarios remain to be fully explored, such as the coincidence of border-collision and Neimark-Sacker bifurcations.

A discontinuity induced bifurcation in a high-dimensional system could be extremely complicated — or it could behave like a bifurcation in low-dimensional system. In the latter case, a more complete understanding of the bifurcation could result from some sort of dimension reduction; however, as of yet no widely applicable theory for dimension reduction exists for PWSC systems.

Resonance tongues in piecewise-linear, continuous maps often have points of zero width (shrinking points) that are located at the intersection of several border-collision curves. While much of the structure of shrinking point bifurcations has been explored, there are still a number of open questions. For example, the theory only applies to shrinking points in rotational symbol sequences; are there similar bifurcations for non-rotational orbits? What differences arise when the lowest order nonlinear terms are not quadratic, but rather some fractional power (as in maps derived from sliding bifurcations in Filippov systems)? Also the observation in [65] that curves of resonance tongue shrinking points may bound chaotic behavior remains to be explained. Finally, at a shrinking point, the map has an invariant polygon that may continue as an invariant circle. This circle can be destroyed by various mechanisms [26, 79, 80]; however, analogies of a number of the rigorous results for the smooth case remain to be obtained for PWSC maps, perhaps due, in part, to a lack of normal hyperbolicity.

## References

- [1] B. Brogliato, *Nonsmooth Mechanics: Models, Dynamics and Control.*, Springer-Verlag, New York, 1999.
- [2] S. Banerjee, G. Verghese (Eds.), *Nonlinear Phenomena in Power Electronics.*, IEEE Press, New York, 2001.
- [3] Z. Zhusubaliyev, E. Mosekilde, *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems.*, World Scientific, Singapore, 2003.

- [4] C. Tse, Complex Behavior of Switching Power Converters., CRC Press, Boca Raton, FL, 2003.
- [5] R. Leine, H. Nijmeijer, Dynamics and Bifurcations of Non-smooth Mechanical systems, volume 18 of *Lecture Notes in Applied and Computational Mathematics*, Springer-Verlag, Berlin, 2004.
- [6] M. di Bernardo, C. Budd, A. Champneys, P. Kowalczyk, Piecewise-smooth Dynamical Systems. Theory and Applications., Springer-Verlag, New York, 2008.
- [7] D. Simpson, Bifurcations in Piecewise-Smooth Continuous Systems., World Scientific, Singapore, 2010.
- [8] S. Rasband, Chaotic Dynamics of Nonlinear Systems., Wiley, New York, 1990.
- [9] P. Glendinning, Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations., Cambridge University Press, New York, 1999.
- [10] H. Nusse, J. Yorke, Border-collision bifurcations for piecewise-smooth one-dimensional maps., *Int. J. Bifurcation Chaos* 5 (1995) 189–207.
- [11] A. Van der Schaft, J. Schumacher, An Introduction to Hybrid Dynamical Systems., Springer-Verlag, New York, 2000.
- [12] M. Johansson, Piecewise Linear Control Systems., volume 284 of *Lecture Notes in Control and Information Sciences*., Springer-Verlag, New York, 2003.
- [13] A. Filippov, Differential Equations with Discontinuous Righthand Sides., Kluwer Academic Publishers., Norwell, 1988.
- [14] R. Leine, D. Van Campen, Bifurcation phenomena in non-smooth dynamical systems., *European J. Mech. A/Solids* 25 (2006) 595–616.
- [15] M. di Bernardo, C. Budd, A. Champneys, Normal form maps for grazing bifurcations in  $n$ -dimensional piecewise-smooth dynamical systems., *Phys. D* 160 (2001) 222–254.
- [16] R. Alur, C. Belta, F. Ivančić, V. Kumar, M. Mintz, G. Pappas, H. Rubin, J. Schug, Hybrid modeling and simulation of biomolecular networks., in: M. Di Benedetto, A. Sangiovanni-Vincentelli (Eds.), *Hybrid Systems: Computation and Control*., volume 2034 of *Lecture Notes in Computer Science*., Springer-Verlag, New York, 2001, pp. 19–32.
- [17] K. Popp, Non-smooth mechanical systems., *J. Appl. Math. Mech.* 64 (2000) 765–772.
- [18] T. Puu, I. Sushko (Eds.), *Business Cycle Dynamics: Models and Tools*., Springer-Verlag, New York, 2006.
- [19] J. Caballé, X. Jarque, E. Michetti, Chaotic dynamics in credit constrained emerging economies., *J. Econ. Dyn. Control* 30 (2006) 1261–1275.
- [20] J. Laugesen, E. Mosekilde, Border-collision bifurcations in a dynamic management game., *Comput. Oper. Res.* 33 (2006) 464–478.
- [21] R. Rosen, *Dynamical System Theory in Biology*., Wiley-Interscience, New York, 1970.
- [22] R. Casey, H. de Jong, J. Gouzé, Piecewise-linear models of genetic regulatory networks: Equilibria and their stability., *J. Math. Biol.* 52 (2006) 27–56.
- [23] X. Zhao, D. Schaeffer, C. Berger, W. Krassowska, D. Gauthier, Cardiac alternans arising from an unfolded border-collision bifurcation., ??? To appear: *J. Comput. Nonlinear Dyn.*
- [24] L. Chua, Chua’s circuit 10 years later., *Internat. J. Circuit Theory Appl.* 22 (1994) 279–305.

- [25] G. Yuan, S. Banerjee, E. Ott, J. Yorke, Border-collision bifurcations in the buck converter., *IEEE Trans. Circuits Systems I Fund. Theory Appl.* 45 (1998) 707–716.
- [26] I. Sushko, L. Gardini, Center bifurcation for a two-dimensional piecewise linear map., 2006. In [18], pages 49–78.
- [27] E. Mosekilde, J. Laugesen, Nonlinear dynamic phenomena in the beer model., *Syst. Dyn. Rev.* 23 (2007) 229–252.
- [28] C. Hommes, H. Nusse, A. Simonovits, Cycles and chaos in a socialist economy., *J. Econ. Dyn. Control* 19 (1995) 155–179.
- [29] W. Chin, E. Ott, H. Nusse, C. Grebogi, Grazing bifurcations in impact oscillators, *Phys. Rev. E* 50 (1994) 4427–4450.
- [30] H. Chang, J. Juang, Piecewise two-dimensional maps and applications to cellular neural networks., *Int. J. Bifurcation Chaos* 14 (2004) 2223–2228.
- [31] S. Coombes, A. Osbaldestin, Period-adding bifurcations and chaos in a periodically stimulated excitable neural relaxation oscillator., *Phys. Rev. E* 62 (2000) 4057–4066.
- [32] D. Simpson, D. Kompala, J. Meiss, Discontinuity induced bifurcations in a model of *Saccharomyces cerevisiae*., *Math. Biosci.* 218 (2009) 40–49.
- [33] M. di Bernardo, C. Budd, A. Champneys, Grazing and border-collision in piecewise-smooth systems: A unified analytical framework., *Phys. Rev. Lett.* 86 (2001) 2553–2556.
- [34] M. Feigin, On the generation of sets of subharmonic modes in a piecewise continuous system., *J. Appl. Math. Mech.* 38 (1974) 759–767.
- [35] H. Lin, P. Antsaklis, Stability and stabilization of switched linear systems: A survey of recent results., *IEEE Trans. Auto. Contr.* 54 (2009) 308–322.
- [36] J. Gonçalves, A. Megretski, M. Dahleh, Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions., *IEEE Trans. Automat. Contr.* 48 (2003) 2089–2106.
- [37] Y. Iwatani, S. Hara, Stability tests and stabilization for piecewise linear systems based on poles and zeros of subsystems., *Automatica* 42 (2006) 1685–1695.
- [38] K. Hadeler, Periodic solutions of homogeneous equations., *J. Diff. Eq.* 95 (1992) 183–202.
- [39] E. Freire, E. Ponce, F. Rodrigo, F. Torres, Bifurcation sets of continuous piecewise linear systems with two zones., *Int. J. Bifurcation Chaos* 8 (1998) 2073–2097.
- [40] V. Carmona, E. Freire, E. Ponce, F. Torres, The continuous matching of two stable linear systems can be unstable, *Disc. Cont. Dyn. Sys.* 16 (2006) 689–703.
- [41] V. Carmona, E. Freire, E. Ponce, F. Torres, Bifurcation of invariant cones in piecewise linear homogeneous systems., *Int. J. Bifurcation Chaos* 15 (2005) 2469–2484.
- [42] T. Küpper, Invariant cones for non-smooth dynamical systems., *Math. Comput. Simulation* 79 (2008) 1396–1408.
- [43] T. Küpper, S. Moritz, Generalized Hopf bifurcation for non-smooth planar systems., *Phil. Trans. R. Soc. Lond. A* 359 (2001) 2483–2496.
- [44] Y. Zou, T. Küpper, Generalized Hopf bifurcation emanated from a corner for piecewise smooth planar systems, *Non. Anal.* 62 (2005) 1–17.

- [45] Y. Zou, T. Küpper, W.-J. Beyn, Generalized Hopf bifurcation for planar Filippov systems continuous at the origin, *J. Nonlinear Sci.* 16 (2006) 159–177.
- [46] Y. Asai, Y. Tasaka, K. Nomura, T. Nomura, M. Casadio, P. Morasso, A model of postural control in quiet standing: Robust compensation of delay-induced instability using intermittent activation of feedback control., *PLoS ONE* 4 (2009) e6169.
- [47] C. Sparrow, Chaos in a three-dimensional single loop feedback system with a piecewise linear feedback function., *J. Math. Anal. Appl.* 83 (1981) 275–291.
- [48] J. Llibre, E. Ponce, A. Teruel, Horseshoes near homoclinic orbits for piecewise linear differential systems in  $\mathbb{R}^3$ ., *Int. J. Bifurcation Chaos* 17 (2007) 1171–1184.
- [49] R. Leine, Bifurcations of equilibria in non-smooth continuous systems., *Phys. D* 223 (2006) 121–137.
- [50] R. Lum, L. Chua, Global properties of continuous piecewise linear vector-fields. 1. Simplest case in  $\mathbb{R}^2$ ., *Int. J. Circ. Theor. Appl.* 19 (1991) 251–307.
- [51] E. Freire, E. Ponce, F. Torres, Hopf-like bifurcations in planar piecewise linear systems, *Publicacions Matemàtiques* 41 (1997) 131–148.
- [52] D. Simpson, J. Meiss, Andronov-Hopf bifurcations in planar, piecewise-smooth, continuous flows., *Phys. Lett. A* 371 (2007) 213–220.
- [53] N. Arima, H. Okazaki, H. Nakano, A generation mechanism of canards in a piecewise linear system., *IEICE Trans. Fund.* E80A (1997) 447–453.
- [54] D. Simpson, J. Meiss, Unfolding a codimension-two discontinuous Andronov-Hopf bifurcation., *Chaos* 18 (2008) 033125.
- [55] Y. Do, S. Kim, P. Kim, Stability of fixed points placed on the border in the piecewise linear systems., *Chaos Solitons Fractals* 38 (2008) 391–399.
- [56] M. Hassouneh, E. Abed, H. Nusse, Robust dangerous border-collision bifurcations in piecewise smooth systems, *Phys. Rev. Lett.* 92 (2004) 070201.
- [57] A. Ganguli, S. Banerjee, Dangerous bifurcation at border collision: When does it occur?, *Phys. Rev. E.* 71 (2005) 057202.
- [58] Y. Do, A mechanism for dangerous border collision bifurcations., *Chaos Solitons Fractals* 32 (2007) 352–362.
- [59] L. Gardini, V. Avrutin, M. Schanz, Connection between bifurcations on the Poincaré equator and the dangerous bifurcations., *Grazer Math. Ber.* 354 (2009) 53–72.
- [60] M. di Bernardo, M. Feigin, S. Hogan, M. Homer, Local analysis of  $C$ -bifurcations in  $n$ -dimensional piecewise-smooth dynamical systems., *Chaos Solitons Fractals* 10 (1999) 1881–1908.
- [61] S. Banerjee, M. Karthik, G. Yuan, J. Yorke, Bifurcations in one-dimensional piecewise smooth maps - Theory and applications in switching circuits., *IEEE Trans. Circuits Systems I Fund. Theory Appl.* 47 (2000) 389–394.
- [62] S. Belghith, Symbolic dynamics in nondifferentiable system originating in R-L-diode driven circuit., *Disc. Cont. Dyn. Sys.* 6 (2000) 275–292.
- [63] I. Sushko, A. Agliari, L. Gardini, Bistability and border-collision bifurcations for a family of unimodal piecewise smooth maps., *Discrete Contin. Dyn. Syst. Ser. B* 5 (2005) 881–897.

- [64] Z. Zhusubaliyev, E. Mosekilde, Torus birth bifurcations in a DC/DC convertor., *IEEE Trans. Circuits Systems I Fund. Theory Appl.* 53 (2006).
- [65] D. Simpson, J. Meiss, Neimark-Sacker bifurcations in planar, piecewise-smooth, continuous maps., *SIAM J. Appl. Dyn. Sys.* 7 (2008) 795–824.
- [66] D. Simpson, J. Meiss, Shrinking point bifurcations of resonance tongues for piecewise-smooth, continuous maps., *Nonlinearity* 22 (2009) 1123–1144.
- [67] D. Simpson, J. Meiss, Resonance near border-collision bifurcations in piecewise-smooth, continuous maps., ??? Submitted to: *Nonlinearity*.
- [68] Z. Zhusubaliyev, E. Mosekilde, S. Maity, S. Mohanan, S. Banerjee, Border collision route to quasiperiodicity: Numerical investigation and experimental confirmation., *Chaos* 16 (2006) 023122.
- [69] W.-M. Yang, B.-L. Hao, How the Arnol'd tongues become sausages in a piecewise linear circle map, *Comm. Theoret. Phys.* 8 (1987) 1–15.
- [70] D. Campbell, R. Galeeva, C. Tresser, D. Uherka, Piecewise linear models for the quasiperiodic transition to chaos., *Chaos* 6 (1996) 121–154.
- [71] I. Sushko, L. Gardini, T. Puu, Tongues of periodicity in a family of two-dimensional discontinuous maps of real Möbius type., *Chaos Solitons Fractals* 21 (2004) 403–412.
- [72] R. Szalai, H. Osinga, Arnol'd tongues arising from a grazing-sliding bifurcation., *SIAM J. Appl. Dyn. Sys.* 8 (2009) 1434–1461.
- [73] R. Szalai, H. Osinga, Invariant polygons in systems with grazing-sliding., *Chaos* 18 (2008) 023121.
- [74] J. Lagarias, E. Rains, Dynamics of a family of piecewise-linear area-preserving plane maps. II. invariant circles., *J. Difference Eqn. Appl.* 11 (2005) 1137–1163.
- [75] Y. Maistrenko, I. Sushko, L. Gardini, About two mechanisms of reunion of chaotic attractors., *Chaos Solitons Fractals* 9 (1998) 1373–1390.
- [76] A. Colombo, F. Dercole, Discontinuity induced bifurcations of non-hyperbolic cycles in nonsmooth systems., *SIAM J. Appl. Dyn. Sys.* 9 (2010) 62–83.
- [77] D. Simpson, J. Meiss, Simultaneous border-collision and period-doubling bifurcations., *Chaos* 19 (2009) 033146.
- [78] X. Zhao, D. Schaeffer, Alternate pacing of border-collision period-doubling bifurcations., *Nonlinear Dyn.* 50 (2007) 733–742.
- [79] Z. Zhusubaliyev, E. Mosekilde, S. De, S. Banerjee, Transitions from phase-locked dynamics to chaos in a piecewise-linear map., *Phys. Rev. E* 77 (2008) 026206.
- [80] Z. Zhusubaliyev, E. Soukhoterlin, E. Mosekilde, Quasiperiodicity and torus breakdown in a power electronic DC/DC converter., *Math. Comput. Simulation* 73 (2007) 364–377.